

ON THE EXISTENCE OF CERTAIN MODULES OF FINITE GORENSTEIN HOMOLOGICAL DIMENSIONS, II

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ABSTRACT. One of the remaining questions in Gorenstein homology is whether a local ring R is Cohen-Macaulay if it possesses a nonzero module which is either finitely generated of finite Gorenstein injective dimension or Cohen-Macaulay of finite G-dimension. In this paper, we continue our investigation on this question. Also, we treat two other closely related questions.

1. INTRODUCTION

Throughout, (R, \mathfrak{m}, k) is a commutative Noetherian local ring. The New Intersection Theorem is one of the most important results in local algebra. It provides simple proofs for several outstanding homological conjectures e.g. Auslander's zero-divisor conjecture [R, Theorem 6.2.3] and Bass' conjecture [PS, Theorem 5.1]. The New Intersection Theorem was proved in prime characteristic by Peskine and Szpiro [PS] in 1973. Then Hochster's works [H1], [H2] established a reduction to prime characteristic from equicharacteristic zero to give a proof of this theorem in all equicharacteristic rings in 1975. Finally, in 1987, Roberts [R] proved the New Intersection Theorem for mixed characteristic rings by using local Chern characters.

The New Intersection Theorem asserts that if

$$0 \longrightarrow F_s \longrightarrow F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

is a non-exact complex of finitely generated free R -modules with finite length homology modules, then $\dim R \leq s$. Using the New Intersection Theorem, one can easily see that if R admits a nonzero Cohen-Macaulay module with $\mathrm{pd}_R M < \infty$, then R must be Cohen-Macaulay. Another implication of the New Intersection Theorem is the Intersection Inequality which says that for any two nonzero finitely generated R -modules M and N if $\mathrm{pd}_R M < \infty$, then

$$\dim_R N \leq \mathrm{pd}_R M + \dim_R(M \otimes_R N).$$

The New Intersection Theorem has another equivalent form which asserts that if

$$0 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \cdots \longrightarrow I_{-s} \longrightarrow 0$$

is a non-exact complex of injective R -modules with finitely generated homology modules, then $\dim R \leq s$; see e.g. [M, page 151, Remark]. Using this, one can easily verify Bass' Theorem to the effect that if R admits a nonzero finitely generated module of finite injective dimension, then R is Cohen-Macaulay.

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Since many results in classical homological algebra have counterparts in Gorenstein homological algebra, the following questions are raised naturally:

Question 1.1. Let (R, \mathfrak{m}, k) be a local ring.

- i) Assume that $0 \rightarrow G_s \rightarrow G_{s-1} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow 0$ is a non-exact complex of totally reflexive R -modules with finite length homology modules. Is $\dim R \leq s$?
- ii) Assume that $0 \rightarrow I_0 \rightarrow I_{-1} \rightarrow \cdots \rightarrow I_{-s} \rightarrow 0$ is a non-exact complex of Gorenstein injective R -modules with finitely generated homology modules. Is $\dim R \leq s$?

Question 1.2. Let (R, \mathfrak{m}, k) be a local ring and M and N two nonzero finitely generated R -modules. Assume that G-dimension of M is finite. Is $\dim_R N \leq \text{G-dim}_R M + \dim_R(M \otimes_R N)$?

Question 1.3. Let (R, \mathfrak{m}, k) be a local ring.

- i) Assume that R admits a nonzero Cohen-Macaulay module of finite G-dimension. Is R Cohen-Macaulay?
- ii) Assume that R admits a nonzero finitely generated module of finite Gorenstein injective dimension. Is R Cohen-Macaulay?

Question 1.3 was asked in [C, pages 40 and 147], [CFoH, Questions 1.31 and 3.26] and [T2]. This question has been studied by many authors; see e.g. [T3], [T2], [FF] and [GHT]. Our investigation on this question was initiated by [DMT]. As we mentioned above, in the proof of the Intersection Inequality, the New Intersection Theorem was used. By [C, Remarks 1.5.7] the answer of Question 1.2 is not positive in general. Hence, one may think that the answer of Question 1.1 is negative, and so it was not asked before. In contrary, we strongly believe that the answer of Question 1.1 is affirmative. It is straightforward to see that an affirmative answer to the second part of Question 1.1 implies an affirmative answer to its first part. Also, note that Question 1.1 easily implies Question 1.3. This paper is concerned with the study of the above questions.

We show that if either R is Cohen-Macaulay or R possesses an R -module B such that $\text{depth}_R B < \infty$ and $\text{depth}_R(B \otimes_R T) \geq \dim R$ for all totally reflexive R -modules T , then the answer of Question 1.1 i) is positive; see Theorem 3.3 below. Also, we prove that if either R is Cohen-Macaulay or R possesses an R -module B such that $\text{width}_R B < \infty$ and $\text{width}_R(\text{Hom}_R(B, T)) \geq \dim R$ for all Gorenstein injective R -modules T , then the answers of Question 1.1 ii) is positive; see Theorem 3.4 below.

By assuming that Question 1.1 i) has an affirmative answer and imposing some extra assumptions on M and N , we provide a positive answer to Question 1.2; see Corollary 4.2 below.

We show that if there exists a big Cohen-Macaulay R -module B such that $\text{Tor}_1^R(B, T) = 0$ for all totally reflexive R -modules T , then the answers of Question 1.3 i) is positive; see Theorem 4.3 below. Also, we prove that if exists a big Cohen-Macaulay R -module B such that $\text{Ext}_R^1(B, T) = 0$ for all Gorenstein injective R -modules T , then the answers of Question 1.3 ii) is positive; see Theorem 4.5 below.

2. PREREQUISITES

Throughout this paper, (R, \mathfrak{m}, k) is a commutative Noetherian local ring with nonzero identity. The \mathfrak{m} -adic completion of an R -module M , will be denoted by \widehat{M} and “complete” stands for “ \mathfrak{m} -adic complete”.

(2.1) Hyperhomology. As we will use technical side of hyperhomology and derived category of R -modules, $\mathcal{D}(R)$, we recall some necessary information and notations which are needed in this paper. The objects in $\mathcal{D}(R)$ are complexes of R -modules and symbol \simeq denotes isomorphisms in this category. For a complex

$$X = \cdots \longrightarrow X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \longrightarrow \cdots$$

in $\mathcal{D}(R)$, its supremum and infimum are defined respectively by $\sup X := \sup\{i \in \mathbb{Z} | H_i(X) \neq 0\}$ and $\inf X := \inf\{i \in \mathbb{Z} | H_i(X) \neq 0\}$, with the usual convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. A complex X is said to be non-exact if it has some nonzero homology modules. The full subcategory of homologically bounded to the right (resp. left) complexes is denoted by $\mathcal{D}_{\square}(R)$ (resp. $\mathcal{D}_{\square}(R)$). Also, the full subcategories of homologically bounded complexes and of complexes with finitely generated homology modules will be denoted by $\mathcal{D}_{\square}(R)$ and $\mathcal{D}^f(R)$, respectively. Modules can be considered as complexes concentrated in degree zero. The full subcategory of homologically bounded complexes with finitely generated homology modules will be denoted by $\mathcal{D}_{\square}^f(R)$.

For any complex X in $\mathcal{D}_{\square}(R)$ (resp. $\mathcal{D}_{\square}(R)$), there is a bounded to the right (resp. left) complex P (resp. I) of projective (resp. injective) R -modules which is isomorphic to X in $\mathcal{D}(R)$. A such complex P (resp. I) is called a projective (resp. injective) resolution of X . By using these resolutions, one can compute derived functors in $\mathcal{D}(R)$. The left derived tensor product functor $-\otimes_R^{\mathbf{L}} \sim$ is computed by taking a projective resolution of the first argument or of the second one. The right derived homomorphism functor $\mathbf{R} \operatorname{Hom}_R(-, \sim)$ is computed by taking a projective resolution of the first argument or by taking an injective resolution of the second one. For any convenient two complexes X, Y and any integer i , set $\operatorname{Tor}_i^R(X, Y) := H_i(X \otimes_R^{\mathbf{L}} Y)$ and $\operatorname{Ext}_R^i(X, Y) := H_{-i}(\mathbf{R} \operatorname{Hom}_R(X, Y))$.

Assume that $X \in \mathcal{D}_{\square}(R)$ and $Y \in \mathcal{D}_{\square}(R)$. The notions $\operatorname{depth}_R X$ and $\operatorname{width}_R Y$ are defined, respectively, by $\operatorname{depth}_R X := -\sup \mathbf{R} \operatorname{Hom}_R(k, X)$ and $\operatorname{width}_R Y := \inf(k \otimes_R^{\mathbf{L}} Y)$. If $X \in \mathcal{D}_{\square}(R)$, [C, Theorem A.6.6] yields that $\operatorname{depth}_R X$ is finite if and only if $\operatorname{width}_R X$ is finite. If $X \in \mathcal{D}_{\square}^f(R)$, then $\operatorname{width}_R X = \inf X$ and $\operatorname{depth}_R X$ is finite if and only if X is non-exact; see respectively [C, A.6.3.2] and [FI, 2.5]. Also, for any $X \in \mathcal{D}(R)$, $\dim_R X$ and $\operatorname{Supp}_R X$ are defined by $\dim_R X := \sup\{\dim R/\mathfrak{p} - \inf X_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec} R\}$ and $\operatorname{Supp}_R X := \{\mathfrak{p} \in \operatorname{Spec} R | X_{\mathfrak{p}} \not\cong 0\}$.

(2.2) Gorenstein homological dimensions. A finitely generated R -module M is said to be *totally reflexive* if there exists an exact complex F of finitely generated free R -modules such that $M \cong \operatorname{im}(F_0 \longrightarrow F_{-1})$ and $\operatorname{Hom}_R(F, R)$ is exact. Obviously, any finitely generated projective module is totally reflexive. Also, an R -module N is said to be *Gorenstein injective* if there exists an exact complex I of injective R -modules such that $N \cong \operatorname{im}(I_1 \longrightarrow I_0)$ and $\operatorname{Hom}_R(E, I)$ is exact for all injective R -modules E . Clearly, any injective module is Gorenstein injective.

For a complex $X \in \mathcal{D}_{\square}^f(R)$, G-dimension of X is defined by

$$\text{G-dim}_R X := \inf\{\sup\{l \in \mathbb{Z} | Q_l \neq 0\} | Q \text{ is a bounded to the right complex of totally reflexive } R\text{-modules such that } Q \simeq X\}.$$

Also, for a complex $Y \in \mathcal{D}_{\square}(R)$, its Gorenstein injective dimension is defined by

$$\text{Gid}_R Y := \inf\{\sup\{l \in \mathbb{Z} | E_{-l} \neq 0\} | E \text{ is a bounded to the left complex of Gorenstein injective } R\text{-modules such that } Y \simeq E\}.$$

3. THE MAIN RESULTS

We start this section with the following lemma which states the Auslander-Buchsbaum and Bass formulas for Gorenstein homological dimensions.

Lemma 3.1. *Let (R, \mathfrak{m}, k) be a local ring and $X \in \mathcal{D}_{\square}^f(R)$.*

- i) *If G-dimension of X is finite, then $\text{G-dim}_R X = \text{depth } R - \text{depth}_R X$.*
- ii) *If Gorenstein injective dimension of X is finite, then $\text{Gid}_R X = \text{depth } R - \inf X$.*

Proof. See [C, Theorem 2.3.13] and [CW, Corollary 2.3]. □

Recall that for an ideal I of R , an R -module M is said to be I -torsion if each element of M is annihilated by some power of I .

Lemma 3.2. *Let (R, \mathfrak{m}, k) be a local ring, M an R -module and $X \in \mathcal{D}_{\square}(R)$.*

- i) *Assume that all homology modules of X are complete. If $\text{Ext}_R^i(M, X) \neq 0$ for some integer i , then $\text{width}_R(\text{Ext}_R^i(M, X)) = 0$.*
- ii) *Assume that all homology modules of X are \mathfrak{m} -torsion. If $\text{Tor}_i^R(M, X) \neq 0$ for some integer i , then $\text{depth}_R(\text{Tor}_i^R(M, X)) = 0$.*

Proof. i) Assume that $\text{Ext}_R^i(M, X) \neq 0$ for some integer i . Let L be a free resolution of M and

$$P = \cdots \longrightarrow P_s \xrightarrow{d_s} P_{s-1} \longrightarrow \cdots \xrightarrow{d_{t+1}} P_t \longrightarrow 0$$

a projective resolution of X . Let \widehat{P} denote the complex

$$\cdots \longrightarrow \widehat{P}_s \xrightarrow{\widehat{d}_s} \widehat{P}_{s-1} \longrightarrow \cdots \xrightarrow{\widehat{d}_{t+1}} \widehat{P}_t \longrightarrow 0.$$

Then [Si, Corollary 5.5] implies that $P \simeq \widehat{P}$, and so by [C, A.4.1], one has:

$$\mathbf{R} \text{Hom}_R(M, X) \simeq \text{Hom}_R(L, X) \simeq \text{Hom}_R(L, \widehat{P}).$$

Every module in the complex $\text{Hom}_R(L, \widehat{P})$ is a product of complete R -modules, and so is complete. Note that by [Si, 1.5] any product of complete R -modules is complete. Since $\text{Ext}_R^i(M, X) \cong H_{-i}(\text{Hom}_R(L, \widehat{P}))$, the claim follows by [Si, Proposition 1.4].

ii) Assume that $\text{Tor}_i^R(M, X) \neq 0$ for some integer i . Let $Y := \text{Hom}_R(X, E(k))$. Then $Y \in \mathcal{D}_{\square}(R)$ and all homology modules of Y are complete by [Si, 4.2]. The isomorphism

$$\text{Hom}_R(\text{Tor}_i^R(M, X), E(k)) \cong \text{Ext}_R^i(M, \text{Hom}_R(X, E(k))) = \text{Ext}_R^i(M, Y)$$

yields that $\text{Ext}_R^i(M, Y) \neq 0$. So, by [FI, Proposition 4.4] and i), we deduce that

$$\begin{aligned} \text{depth}_R(\text{Tor}_i^R(M, X)) &= \text{width}_R(\text{Hom}_R(\text{Tor}_i^R(M, X), E(k))) \\ &= \text{width}_R(\text{Ext}_R^i(M, Y)) \\ &= 0. \end{aligned}$$

□

The following result is one of the main results of this paper. It provides some partial answers to Question 1.1 i). Before presenting it, we recall the Right Acyclicity Lemma for the right connected sequence of functors $\text{Ext}_R^i(k, \cdot)$ and the Left Acyclicity Lemma for the left connected sequence of functors $\text{Tor}_i^R(k, \cdot)$.

Let

$$Y := 0 \longrightarrow Y_s \longrightarrow Y_{s-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0$$

be a complex of R -modules such that for each $i = 1, \dots, s$, both $\text{depth}_R Y_i \geq i$ and either $H_i(Y) = 0$ or $\text{depth}_R H_i(Y) = 0$. Then the Right Acyclicity Lemma [St, Proposition 1.1.1] asserts that Y is exact. Now, let

$$X := X_0 \longrightarrow X_{-1} \longrightarrow \cdots \longrightarrow X_{-s+1} \longrightarrow X_{-s} \longrightarrow 0$$

be a complex of R -modules such that for each $i = 1, \dots, s$, both $\text{width}_R X_{-i} \geq i$ and either $H_{-i}(X) = 0$ or $\text{width}_R H_{-i}(X) = 0$. Then the Left Acyclicity Lemma [St, Proposition 1.1.2] asserts that X is exact.

Theorem 3.3. *Let (R, \mathfrak{m}, k) be a local ring and $G = 0 \longrightarrow G_s \longrightarrow G_{s-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow 0$ a non-exact complex of totally reflexive R -modules with finite length homology modules. If either:*

- i) *there exists an R -module B such that $\text{depth}_R B < \infty$ and $\text{depth}_R(B \otimes_R M) \geq \dim R$ for all totally reflexive R -modules M ; or*
- ii) *R is Cohen-Macaulay,*

then $\dim R \leq s$.

Proof. i) By [C, Theorem A.6.6], we see that $\text{width}_R B < \infty$. Set $t := \inf G$ and $h := \text{width}_R B$. By [C, Theorem A.3.2 L)], one can choose a resolution

$$F := \cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_{t+1} \longrightarrow F_t \longrightarrow 0$$

of finitely generated free R -modules such that $F \simeq G$. Since $\text{G-dim}_R G \leq s$, [C, Theorem 2.3.7] yields that $C_n^F := \text{coker}(F_{n+1} \longrightarrow F_n)$ is a totally reflexive R -module for all $n \geq s$. By [C, Lemma A.6.5 and A.6.3.2],

$$\begin{aligned} \text{width}_R(B \otimes_R^{\mathbf{L}} G) &= \text{width}_R B + \text{width}_R G \\ &= \text{width}_R B + \inf G \\ &= h + t. \end{aligned}$$

Now,

$$\begin{aligned}
 h + t &= \text{width}_R(B \otimes_R^{\mathbf{L}} G) \\
 &\geq \inf k + \inf(B \otimes_R^{\mathbf{L}} G) \\
 &\geq \inf B + \inf G \\
 &= t.
 \end{aligned}$$

Thus $l := \inf(B \otimes_R^{\mathbf{L}} G)$ is an integer, and so the complex

$$Y := 0 \longrightarrow C_{l+s}^{B \otimes_R F} \longrightarrow B \otimes_R F_{l+s-1} \longrightarrow \cdots \longrightarrow B \otimes_R F_{l+1} \longrightarrow B \otimes_R F_l \longrightarrow B \otimes_R F_{l-1}$$

is not exact. By Lemma 3.2 ii), any nonzero homology module of Y is of zero depth. Set $Y_i := B \otimes_R F_{l+i-1}$ for $i = 0, \dots, s$ and $Y_{s+1} := C_{l+s}^{B \otimes_R F}$. Now, by applying the Right Acyclicity Lemma (see paragraph preceding this theorem) to the complex Y , we conclude that $\text{depth}_R Y_i < i \leq s + 1$ for some positive integer i . By our assumption on B , we have

$$\text{depth}_R Y_i = \text{depth}_R(B \otimes_R F_{l+i-1}) \geq \dim R$$

for all $0 \leq i \leq s$ and

$$\text{depth}_R Y_{s+1} = \text{depth}_R C_{l+s}^{B \otimes_R F} = \text{depth}_R(B \otimes_R C_{l+s}^F) \geq \dim R.$$

Thus $\dim R \leq s$.

ii) By Lemma 3.1 i), $\text{depth}_R M = \text{depth } R$ for all nonzero totally reflexive R -modules M . Hence, the assertion is clear by letting $B := R$ and using i). \square

In the next result, we present a partial answer to Question 1.1 ii). It also provides a dual of Theorem 3.3.

Theorem 3.4. *Let (R, \mathfrak{m}, k) be a local ring and $I = 0 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \cdots \longrightarrow I_{-s} \longrightarrow 0$ a non-exact complex of Gorenstein injective R -modules with finitely generated homology modules. If either:*

- i) *there exists an R -module B such that $\text{width}_R B < \infty$ and $\text{width}_R(\text{Hom}_R(B, N)) \geq \dim R$ for all Gorenstein injective R -modules N ; or*
- ii) *R is Cohen-Macaulay,*

then $\dim R \leq s$.

Proof. i) By [FF, Theorem 3.6], $\text{Gid}_{\widehat{R}}(I \otimes_R \widehat{R})$ is finite. So, Lemma 3.1 ii) yields that

$$\begin{aligned}
 \text{Gid}_{\widehat{R}}(I \otimes_R \widehat{R}) &= \text{depth } \widehat{R} - \inf(I \otimes_R \widehat{R}) \\
 &= \text{depth } R - \inf I \\
 &= \text{Gid}_R I \\
 &\leq s.
 \end{aligned}$$

Hence, by [C, Theorem A.3.2] and [CFrH, Theorem 3.3], there exists a bounded complex

$$Y := 0 \longrightarrow Y_g \longrightarrow Y_{g-1} \longrightarrow \cdots \longrightarrow Y_{-t} \longrightarrow 0$$

of Gorenstein injective \widehat{R} -modules of length at most $s + 1$ such that $Y \simeq I \otimes_R \widehat{R}$. So, Y is non-exact and all of its homolog modules are finitely generated \widehat{R} -modules. Also, by [ND, Lemma 2.4] any Gorenstein injective \widehat{R} -module is also Gorenstein injective as an R -module. On the other hand, $\text{width}_{\widehat{R}}(B \otimes_R \widehat{R}) < \infty$ and for any Gorenstein injective \widehat{R} -module N , we have:

$$\begin{aligned} \text{width}_{\widehat{R}}(\text{Hom}_{\widehat{R}}(B \otimes_R \widehat{R}, N)) &= \text{width}_{\widehat{R}}(\text{Hom}_R(B, \text{Hom}_{\widehat{R}}(\widehat{R}, N))) \\ &= \text{width}_{\widehat{R}}(\text{Hom}_R(B, N)) \\ &= \text{width}_R(\text{Hom}_R(B, N)). \end{aligned}$$

Thus, we may and do assume that R is complete.

By [FI, 2.5], $\text{depth}_R I$ is finite. Hence [C, Lemma A.6.4] yields that

$$\text{depth}_R(\mathbf{R} \text{Hom}_R(B, I)) = \text{width}_R B + \text{depth}_R I < \infty,$$

and so the complex $\mathbf{R} \text{Hom}_R(B, I)$ is not exact. Let J be an injective resolution of I such that $J_i = 0$ for all $i > \sup I$. As $\text{Gid}_R I \leq s$, [CFrH, Theorem 3.3] implies that $Z_{-n}^J := \ker(J_{-n} \rightarrow J_{-n-1})$ is a Gorenstein injective R -module for all $n \geq s$. Since $\mathbf{R} \text{Hom}_R(B, I) \simeq \text{Hom}_R(B, J)$, it follows that the complex $\text{Hom}_R(B, J)$ is not exact. Set $l := \sup(\text{Hom}_R(B, J))$. Then [C, Proposition A.4.6] implies that $l \leq \sup I$. Hence l is an integer, and so the complex

$$X := \text{Hom}_R(B, J_{l+1}) \rightarrow \text{Hom}_R(B, J_l) \rightarrow \cdots \rightarrow \text{Hom}_R(B, J_{l-s+1}) \rightarrow Z_{l-s}^{\text{Hom}_R(B, J)} \rightarrow 0$$

is not exact. Set $X_{-i} := \text{Hom}_R(B, J_{l+1-i})$ for $i = 0, \dots, s$ and $X_{-s-1} := Z_{l-s}^{\text{Hom}_R(B, J)}$. Since R is complete, the homology modules of I , and so that of J are complete. So, Lemma 3.2 i) yields that the width of any nonzero homology module of X is zero. Now, by applying the Left Acyclicity Lemma (see paragraph preceding Theorem 3.3) on the complex X , we deduce that $\text{width}_R X_{-i} < i \leq s + 1$ for some positive integer i . By the assumption,

$$\text{width}_R X_{-i} = \text{width}_R(\text{Hom}_R(B, J_{l+1-i})) \geq \dim R$$

for all $0 \leq i \leq s$ and

$$\text{width}_R X_{-s-1} = \text{width}_R Z_{l-s}^{\text{Hom}_R(B, J)} = \text{width}_R(\text{Hom}_R(B, Z_{l-s}^J)) \geq \dim R.$$

Thus $\dim R \leq s$.

ii) Suppose that R is Cohen-Macaulay. By [CW, Lemma 2.1], $\text{width}_R N \geq \dim R$ for all Gorenstein injective R -modules N . So, the assertion follows by letting $B := R$ in i). \square

Now, we provide big Cohen-Macaulay R -modules satisfy the assumptions of Theorems 3.3 i) and 3.4 i). Recall that an R -module B is called *big Cohen-Macaulay* if there exists a system of parameters $x := x_1, \dots, x_n$ such that x is a B -regular sequence. Hochster [H2] has proved that if R is equicharacteristic (i.e. R and k have the same characteristic), then it admits a big Cohen-Macaulay module.

Lemma 3.5. *Let (R, \mathfrak{m}, k) be a local ring and B a big Cohen-Macaulay R -module. Assume that M is a totally reflexive R -module and N a Gorenstein injective R -module.*

i) *If $\text{Tor}_1^R(B, T) = 0$ for all totally reflexive R -modules T , then $\text{depth}_R(M \otimes_R B) \geq \dim R$.*

ii) If $\text{Ext}_R^1(B, T) = 0$ for all Gorenstein injective R -modules T , then $\text{width}_R(\text{Hom}_R(B, N)) \geq \dim R$.

Proof. Since i) and ii) have similar proofs, we only prove i).

Set $d := \dim R$. We can assume that M is nonzero. Since M is totally reflexive, there exists an exact sequence

$$F := 0 \longrightarrow M \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots$$

in which each F_i is a finitely generated free R -module and each $K_{-i} := \text{im}(F_{-i} \longrightarrow F_{-(i+1)})$ is a totally reflexive R -module. By the assumption, we have the following short exact sequences

$$0 \longrightarrow M \otimes_R B \longrightarrow F_0 \otimes_R B \longrightarrow K_0 \otimes_R B \longrightarrow 0$$

and

$$0 \longrightarrow K_{1-i} \otimes_R B \longrightarrow F_{-i} \otimes_R B \longrightarrow K_{-i} \otimes_R B \longrightarrow 0$$

for all $i \geq 1$.

Now, we have two cases. At first, we assume that there exists a non-negative integer i such that

$$d < \text{depth}_R(K_{-i} \otimes_R B) + 1,$$

and we set $s := \inf\{j \in \mathbb{N}_0 \mid d < \text{depth}_R(K_{-j} \otimes_R B) + 1\}$. Since $\text{depth}_R B = \dim R$, by applying [BH, Proposition 1.2.9] on the above short exact sequences successively, we deduce that

$$\begin{aligned} \text{depth}_R(M \otimes_R B) &\geq \min\{d, \text{depth}_R(K_0 \otimes_R B) + 1\} \\ &\geq \min\{d, \text{depth}_R(K_{-1} \otimes_R B) + 1\} + 1 \\ &\vdots \\ &\geq \min\{d, \text{depth}_R(K_{-s} \otimes_R B) + 1\} + s \\ &\geq d. \end{aligned}$$

So, the proof is complete in this case. Now, assume that

$$\text{depth}_R(K_{-i} \otimes_R B) + 1 \leq d$$

for all $i \in \mathbb{N}_0$. By repeating the above inequalities d times, the assertion follows. \square

Remark 3.6. i) By Lemma 3.5, certain big Cohen-Macaulay R -modules satisfy the assumptions of part i) of Theorems 3.3 and 3.4. Now, let B be as in part i) of either Theorem 3.3 or Theorem 3.4. Then using [BH, page 353, Ex. 9.1.12], it is easy to check that \widehat{B} is a big Cohen-Macaulay R -module.

ii) In view of Lemma 3.5, one may guess that our assumptions on the module B in part i) of Theorems 3.3 and 3.4 are satisfied by any big Cohen-Macaulay R -module. But, this is not the case. To this end, let $R := k[[X, Y]]/(XY)$, where k is a field. Then R is a Gorenstein complete local ring of dimension one. Denote by x and y the residue classes of X and Y in R and set $B := R/(x)$ and $M := R/(y)$. We can easily check that B and M are maximal Cohen-Macaulay R -modules, and so both B and M are totally reflexive. Now, as $B \otimes_R M \cong k$,

we have $\text{depth}_R(B \otimes_R M) = 0$. Also, by [C, Theorems 5.1.11 and 6.4.2], $N := \text{Hom}_R(M, E(k))$ is a Gorenstein injective R -module and

$$\begin{aligned} \text{width}_R(\text{Hom}_R(B, N)) &= \text{width}_R(\text{Hom}_R(B, \text{Hom}_R(M, E(k)))) \\ &= \text{width}_R(\text{Hom}_R(B \otimes_R M, E(k))) \\ &= \text{width}_R(\text{Hom}_R(k, E(k))) \\ &= \text{width}_R k \\ &= 0. \end{aligned}$$

- iii) One may conjecture that if an R -module B is such that $\text{depth}_R B < \infty$ and $\text{depth}_R(M \otimes_R B) \geq \dim R$ for all totally reflexive R -modules M , then either $B = R$ or the class of totally reflexive R -modules coincides with the class of finitely generated free R -modules. Similarly, one may guess that if an R -module B is such that $\text{width}_R B < \infty$ and $\text{width}_R(\text{Hom}_R(B, N)) \geq \dim R$ for all Gorenstein injective R -modules N , then either $B = R$ or the class of Gorenstein injective R -modules coincides with the class of injective R -modules. But, this is not the case. To realize this, let (R, \mathfrak{m}, k) be an Artinian Gorenstein local ring which is not regular. Then, the R -module k is both totally reflexive and Gorenstein injective, while it is neither free nor injective. On the other hand, any R -module with finite depth can be chosen as B .
- iv) Let (R, \mathfrak{m}, k) be an Artinian Gorenstein local ring which is not regular. Then the R -module k is both totally reflexive and Gorenstein injective. Set $B := k$. Then B is a big Cohen-Macaulay R -module and $\text{depth}_R(B \otimes_R M) \geq \dim R$ (resp. $\text{width}_R(\text{Hom}_R(B, N)) \geq \dim R$) for all totally reflexive (resp. Gorenstein injective) R -modules M (resp. N). As k is neither free nor injective, we deduce that $\text{Tor}_1^R(B, k) \neq 0$ and $\text{Ext}_R^1(B, k) \neq 0$. Thus, in Lemma 3.5, the converse of neither part i) nor part ii) holds.

4. APPLICATIONS

We begin this section by showing that the Gorenstein analogue of the New Intersection Theorem implies the Gorenstein analogue of the Intersection Inequality in a special case.

Theorem 4.1. *Let (R, \mathfrak{m}, k) be a local ring, N a nonzero cyclic R -module and $X \in \mathcal{D}_\square^f(R)$ a non-exact complex. Assume that $\text{pd}_R N < \infty$, $\text{G-dim}_R X < \infty$ and $\text{Supp}_R(X \otimes_R^{\mathbf{L}} N) = \{\mathfrak{m}\}$. If Question 1.1 i) has a positive answer (e.g. R is Cohen-Macaulay), then*

$$\dim_R N \leq \text{G-dim}_R X + \dim_R(X \otimes_R^{\mathbf{L}} N).$$

Proof. Set $s := \text{G-dim}_R X$ and $t := \inf X$. By [C, Theorem 2.3.7], there exists a complex

$$G := 0 \longrightarrow G_s \longrightarrow G_{s-1} \longrightarrow \cdots \longrightarrow G_{t+1} \longrightarrow G_t \longrightarrow 0$$

of totally reflexive R -modules such that $G \simeq X$.

At first, we let $N = R$. As $\text{Supp}_R X = \{\mathfrak{m}\}$, it follows that $\inf X_{\mathfrak{p}} = \infty$ for all $\mathfrak{p} \in \text{Spec } R - \{\mathfrak{m}\}$, and so $\dim_R X = -\inf X$. Since the complex G satisfies the assumptions of Question 1.1 i), one obtains

$$\dim R \leq s - t = \text{G-dim}_R X + \dim_R X.$$

Now, assume that N is an arbitrary cyclic R -module. Then there exists an ideal I of R such that $N \cong R/I$. Consider the natural epimorphism $R \rightarrow R/I$. Set $X' := X \otimes_R^{\mathbf{L}} R/I$. Then $X' \in \mathcal{D}_{\square}^f(R/I)$, and one can easily check that $\text{Supp}_{R/I} X' = \{\mathfrak{m}/I\}$ and $\dim_R X' = \dim_{R/I} X'$. Since $\text{pd}_R R/I$ is finite, [C, Theorem 5.1.11] and [CFrH, Corollary 2.16] imply that

$$X' \simeq G \otimes_R^{\mathbf{L}} R/I \simeq G \otimes_R R/I.$$

By [CFoH, Proposition 1.5], it turns out that $G \otimes_R R/I$ consists of totally reflexive R/I -modules. So, $\text{G-dim}_{R/I} X' \leq s$ and then the first part of the proof implies that

$$\dim R/I \leq \text{G-dim}_{R/I} X' + \dim_{R/I} X' \leq s + \dim_R (X \otimes_R^{\mathbf{L}} N),$$

as desired. \square

Corollary 4.2. *Let (R, \mathfrak{m}, k) be a local ring, N a nonzero cyclic R -module and M a nonzero finitely generated R -module. Assume that $\text{pd}_R N < \infty$, $\text{G-dim}_R M < \infty$ and $l_R(M \otimes_R N) < \infty$. If Question 1.1 i) has a positive answer (e.g. R is Cohen-Macaulay), then $\dim_R N \leq \text{G-dim}_R M$.*

Proof. By [F, Proposition 3.5], one has

$$\dim_R(M \otimes_R^{\mathbf{L}} N) = \sup\{\dim_R(\text{Tor}_R^i(M, N)) - i \mid i \in \mathbb{N}_0\} = \dim_R(M \otimes_R N).$$

So, the assertion follows by setting $X := M$ in Theorem 4.1. \square

Now, as an application of Theorem 3.3 i), we give a partial answer to Question 1.3 i). We recall that every local ring containing a field has a big Cohen-Macaulay module.

Theorem 4.3. *Let (R, \mathfrak{m}, k) be a local ring. Assume that there exists a nonzero Cohen-Macaulay R -module M with finite G -dimension. The following are equivalent:*

- i) R is a Cohen-Macaulay ring.
- ii) There exists a big Cohen-Macaulay R -module B such that $\text{Tor}_1^R(B, T) = 0$ for all totally reflexive R -modules T .
- iii) There exists an R -module B such that $\text{depth}_R B < \infty$ and $\text{depth}_R(B \otimes_R T) \geq \dim R$ for all totally reflexive R -modules T .

Proof. Suppose that $x_1, \dots, x_t \in \mathfrak{m}$ is a maximal M -regular sequence. Then the R -module $M/(x_1, \dots, x_t)M$ has finite length and [A, Theorem 8.7.7]) implies that

$$\text{G-dim}_R M/(x_1, \dots, x_t)M = \text{G-dim}_R M + t.$$

Hence, we may and do assume that M has finite length.

$i \Rightarrow ii$) Take $B := R$.

$ii \Rightarrow iii$) It follows from Lemma 3.5 i). Note that as $B/\mathfrak{m}B \neq 0$, we have $\text{width}_R B < \infty$, and so [C, Theorem A.6.6] implies that $\text{depth}_R B < \infty$.

$iii \Rightarrow i$) Set $s := \text{depth}_R R$. Lemma 3.1 i) implies that $\text{G-dim}_R M = s$, and so there is a non-exact complex

$$G := 0 \rightarrow G_s \rightarrow \dots \rightarrow G_0 \rightarrow 0$$

of totally reflexive R -modules such that $G \simeq M$. Now, by Theorem 3.3 i), it turns out that $\dim R \leq s$, and so R is Cohen-Macaulay. \square

The following example shows that the conditions of Theorem 4.3 are not equivalent in general.

Example 4.4. Let $R := k[[X, Y]]/(X^2, XY)$, where k is a field. By [T1, Example 5.5], the two classes of totally reflexive R -modules and finitely generated free R -modules are the same. Set $B := R/(x)$, where x is the residue class of X in R . Then B is a maximal Cohen-Macaulay R -module. It is trivial that $\mathrm{Tor}_1^R(B, T) = 0$ for all totally reflexive R -modules T , while R is not a Cohen-Macaulay ring.

We end this paper by the following result which gives a partial answer to Question 1.3 ii).

Theorem 4.5. *Let (R, \mathfrak{m}, k) be a local ring. Assume that there exists a nonzero finitely generated R -module N with finite Gorenstein injective dimension. The following are equivalent:*

- i) R is a Cohen-Macaulay ring.
- ii) There exists a big Cohen-Macaulay R -module B such that $\mathrm{Ext}_R^1(B, T) = 0$ for all Gorenstein injective R -modules T .
- iii) There exists an R -module B such that $\mathrm{width}_R B < \infty$ and $\mathrm{width}_R(\mathrm{Hom}_R(B, T)) \geq \dim R$ for all Gorenstein injective R -modules T .

Proof. $i \Rightarrow ii$) Take $B := R$.

$ii \Rightarrow iii$) It follows by Lemma 3.5 ii). Note that as $B/\mathfrak{m}B \neq 0$, we have $\mathrm{width}_R B < \infty$.

$iii \Rightarrow i$) Set $s := \mathrm{depth} R$. Lemma 3.1 ii) implies that $\mathrm{Gid}_R N = s$, and so there is a non-exact complex

$$I := 0 \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_{-s} \longrightarrow 0$$

of Gorenstein injective R -modules such that $N \simeq I$. Now, by Theorem 3.4 i), it turns out that $\dim R \leq s$, and so R is Cohen-Macaulay. \square

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